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On the singularity structure and stability of plane waves

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Abstract: We describe various aspects of plane wave backgrounds. In particular, we make explicit a simple criterion for singularity by establishing a relation between Brinkmann metric entries and diffeomorphism-invariant curvature information. We also address the stability of plane wave backgrounds by analyzing the fluctuations of generic scalar modes. We focus our attention on cases where after fixing the light-cone gauge the resulting world sheet fields appear to have negative "mass terms". We nevertheless argue that these backgrounds may be stable.

Keywords: Black Holes in String Theory, Penrose limit and pp-wave background.

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1. Introduction and summary

Plane wave spacetimes have special properties that motivate their study in both General Relativity and string theory. Due to the presence of a covariantly constant null Killing vector and the specific structure of the curvature invariants, it was suggested early on that α' corrections were under control for stringy plane wave backgrounds. Along these lines some concrete realizations were found in the context of WZW models providing examples of exact strings backgrounds on curved spacetimes [1]–[16]. More recently, renewed interest in plane wave backgrounds has resulted from the realization that certain plane waves are maximally supersymmetric solutions of eleven dimensional and IIB supergravity [17, 18] (see also [19]). More remarkable is the fact that some of these backgrounds allow for exact quantization in the light-cone gauge [20] (see also [21]), thus providing examples of tractable string theory backgrounds with Ramond-Ramond (RR) fluxes on curved spacetimes. Finally, based on the fact that some of the maximally supersymmetric plane waves can be obtained as Penrose-Güven limits [22, 23] of $AdS_p \times S^q$ backgrounds [24], a gauge theory interpretation based on the AdS/CFT correspondence was proposed in [25].

These special properties have motivated the study of many generalizations. One particularly prolific direction is the study of Penrose limits of known supergravity backgrounds. The Penrose limit, generalized to include form fields and fermions by Güven, yields a plane wave metric. Some of these metrics are known, however many present new and unexpected features. This situation motivated us to study a fairly general class of such backgrounds.

A particular class of interest are metrics which in the string frame have the so-called Brinkmann [26] form

$$ds^2 = -4dx^+dx^- - (x^i\mu_{ij}^2 x^j)(dx^+)^2 + \delta_{ij}dx^i dx^j \, , \tag{1.1}$$

where the matrix μ_{ij}^2 is an arbitrary function of x^+. Any other fields should depend only on the coordinate x^+. Furthermore, the field strengths of any gauge fields should annihilate the vector field $\frac{\partial}{\partial x^-}$. It is clear that the x^- translation $x^- \to x^- + \epsilon$ with all other coordinates held constant is a symmetry, so that $\frac{\partial}{\partial x^-}$ is a (null) Killing field.

We are interested in two very natural, though separate, questions that arise in the context of Penrose limits of string theory backgrounds: (i) When is a plane wave metric singular? (ii) When is a plane wave background stable? Since scalar curvature invariants vanish for plane waves, it is useful to identify diffeomorphism invariant characterizations of the properties of (1.1). In section 2 we make explicit a criterion for detecting curvature singularities in plane wave backgrounds by relating a diffeomorphism invariant object to the structure of the metric. Our criterion provides an immediate answer to the question of the singularity structure.

Our second question concerns stability. The case where μ_{ij}^2 has at least one negative eigenvalue is interesting because this seems to imply the existence of worldsheet tachyonic fields in light-cone gauge. Even though this is a gauge-dependent statement there are some interesting effects. Physically, such negative eigenvalues of μ_{ij}^2 imply that objects are stretched and separated by gravitational tidal forces. The effect is similar to that felt in de Sitter space, though without the formation of horizons. Here we are primarily concerned with stability in non-singular (smooth) backgrounds. In fact, since the question of stability is most cleanly formulated in time-independent settings, we focus mainly on the case where μ_{ij}^2 is independent of x^+.

We begin to address stability in section 3, considering a scalar mode as a typical example of a perturbation. While we do not conclusively demonstrate either stability or instability, our main point is to show that the light-cone gauge tachyons do not by themselves indicate instability. We analyze the dispersion relation in two regimes and show that, although the squared-mass of a string state becomes negative in one of these regimes, the corresponding modes have real time evolution and are stable. Thus, the behavior is markedly different from that of a field with constant negative mass-squared. In the end, we reduce the problem of stability to a rather technical one involving the completeness of our set of modes.

Finally, we address the perturbative stability of quantum fields (sections 3.3, 3.4) and of string theory at the level of free strings (section 4). Again, we focus on homogeneous plane wave backgrounds (constant μ_{ij}^2). As in the classical setting, we make only first steps toward analyzing stability, but we show that the light-cone tachyons do not by themselves lead to instability. In the discussion section we point out a number of open issues concerning string interactions. We refer the reader to [28] for a comments on the classical stability analysis of other supergravity fields. Our work has a certain overlap with that of [28], though the results were obtained independently and simultaneously.

2. Plane waves: singular or regular

In order to arrive at a criterion for regularity we need to describe some aspects of the geometry of plane waves. We proceed quickly since most of these results are at least implicitly discussed in the literature (especially in the classic works [26, 29, 30, 31, 32, 33]). Nevertheless, it is useful to have such results in explicit form.

2.1 Symmetries

Any metric of the Brinkmann form (1.1) has a rather high degree of symmetry. Killing's equation is:

$$0 = \xi_{\mu;\nu} + \xi_{\nu;\mu} = \partial_\nu \xi_\mu + \partial_\mu \xi_\nu - 2\Gamma^\alpha_{\nu\mu}\xi_\alpha \,, \tag{2.1}$$

and the only nonzero Christoffel symbols (with all indices down) are

$$\begin{aligned} \Gamma_{++i} &= -\mu^2_{ij}x^j = -\Gamma_{i++} \,, \\ \Gamma_{+++} &= -\frac{1}{2}x^i \left(\frac{\partial}{\partial x^+}\mu^2_{ij}\right) x^j \,, \end{aligned} \tag{2.2}$$

We use the conventions of Misner, Thorne, and Wheeler [34].

Consider a Killing field with $\xi_- = 0$ and with ξ_i a function only of x^+. A lengthy but straightforward calculation shows that all Killing fields in fact satisfy the first of these conditions while the second condition holds for all Killing fields except for possible rotational symmetries of μ_{ij}. Given these conditions, Killing's equation reduces to

$$\partial_+ \xi_+ = x^i \mu^2_{ij}\xi^j \,, \qquad \text{and} \qquad \partial_+\xi_i + \partial_i \xi_+ = 0 \,. \tag{2.3}$$

Latin indices are freely raised and lowered with the flat euclidean metric δ_{ij}. Combining the above two equations yields

$$\partial^2_+ \xi_i + \mu^2_{ij}\xi^j = 0 \,, \tag{2.4}$$

which has solutions for all initial data $\xi^i(0)$, $\partial_+\xi^i(0)$. The general solution for ξ_+ is then given by

$$\xi_+ = -x^i \partial_+ \xi_i + c \,, \tag{2.5}$$

where c is an arbitrary constant.

Note that we have identified enough Killing fields to show that each null surface $x^+ =$ constant is a homogeneous surface. Perhaps a more direct way to realize this is by transforming to Rosen, or group, coordinates (see e.g. [35]):

$$v = x^- + 2\dot{q}_{ab}(x^+)X^a X^b \,, \qquad x^a = P^a_b(x^+)X^b \,, \tag{2.6}$$

with $q_{ab} = P^c_a P^c_b$ and

$$\ddot{P}^a_b(x^+) = H_{ac}(x^+)P^a_c(x^+) \tag{2.7}$$

with, e.g., $P^a_b(0) = \delta^a_b$ and $\dot{P}^a_b(0) = 0$. In such coordinates, the metric takes the form

$$ds^2 = -4dx^+dx^- + q_{ab}(x^+)dX^a dX_b \,. \tag{2.8}$$

The detailed properties of the Killing fields depend on the choice of initial data $\xi^i(0), \partial_+\xi^i(0)$. For definiteness, let us consider the time independent case, where ∂_+ is also a Killing field. Then the ξ^i form a collection of harmonic or inverted harmonic oscillators. Some choices of initial data thus lead to 'corkscrew' motions which wind or rotate around the origin. Such Killing fields are no doubt related to the rotating D-branes of [36].

2.2 Geodesics and curvatures

Let us now consider the curve $x^i = 0, x^- = 0$. Since the metric (1.1) is symmetric under the reflection $(x^i, x^-, x^+) \to (-x^i, -x^-, -x^+)$, an acceleration vector for this curve at $x^+ = 0$ must be invariant under this symmetry. As a result, this vector can point only along the curve. But since the origin of x^+ is arbitrary, we see that the curve is a geodesic. Direct calculation then shows that x^+ is an associated affine parameter.

A large family of null geodesics can be obtained by applying symmetries to the curve $x^i = 0, x^- = 0$; i.e., by acting with the translation e^ξ. These geodesics satisfy $x^+ = \lambda$, $x^- = \frac{1}{4}(\xi^i\partial_+\xi_i) - c$, and $x^i(x^+) = \xi^i(x^+)$ with λ an affine parameter and ξ some Killing field of the form discussed in section 2.1. The fact that these curves trace out geodesics explains why equation (2.4) will look familiar to readers who have not previously studied the Killing symmetries of these spacetimes but who have studied geodesic motion. This family contains all of the null geodesics except those that follow the orbits of $\frac{\partial}{\partial x^-}$.

Suppose that two geodesics are related by a flow through a unit Killing parameter along some Killing field ξ of the form described in section 2.1. Then the distance separating these geodesics along any surface $x^+ =$ constant is given by $|\xi| = \sqrt{|\sum_i \xi^i\xi^i|^2}$. Since x^+ is an affine parameter for this family of geodesics, it is also meaningful to speak of the relative velocity $\partial_+\xi^\alpha$ and acceleration $\partial_+^2\xi^\beta$ of these geodesics, so long as we keep in mind the freedom to rescale any affine parameter by an overall (i.e., x^+ independent) constant and to shift its origin. In particular, this information is diffeomorphsim invariant modulo the potential scaling and shift. Note that these relative accelerations measure gravitational tidal forces and, if the geodesics are infinitesimally separated, they are given by certain components of the Riemann tensor:

$$\partial_+^2\xi^\nu = -R^\nu_{\alpha\beta\gamma}\xi^\beta t^\alpha t^\gamma\,, \tag{2.9}$$

where t^α is the tangent vector to the geodesic. If we take one of the geodesics to be at $x^i = 0$, then this tangent vector is just $t^\alpha\partial_\alpha = \frac{\partial}{\partial x^+}$.

Thus for Killing vector fields ξ and ξ' we have

$$\xi^i\mu^2_{ij}\xi'^j = -\xi_i\partial_+^2\xi'^i = \xi_\nu R^\nu_{\alpha\beta\gamma}\xi'^\beta t^\alpha t^\gamma\,, \tag{2.10}$$

so that the matrix-valued function $\mu^2_{ij}(x^+) = R_{i+j+}(x^+)$ describes diffeomorphism-invariant curvature information (up to an overall scale and shift of the origin of x^+).

Suppose then that some component of μ^2_{ij} diverges at some finite value of x^+. It is evident that invariantly defined components of the curvature tensor diverge[1] at this surface.

[1]The diverging tidal forces are sufficient to overcome any internal forces within an object approaching

Since x^+ is an affine parameter along the null geodesic at the origin, it is also evident that geodesics reach this singularity in finite affine parameter. Thus, such spacetimes contain curvature singularities.

3. Field theory on plane waves with negative eigenvalues

In the appendix we review a particularly interesting Penrose limit, starting from an AdS-Schwarzschild black hole and basing the limit on a geodesic which falls into the singularity. One of the intriguing features of this background (A.20) is the negative eigenvalue (associated with x_8) of the matrix μ_{ij}^2 that arises for small r. If one attempts to quantize string theory in the light-cone gauge on the background (A.20) one would have a negative "mass term" for the worldsheet boson x_8. One might wonder whether this is a tachyonic instability. The appearance of these negative "mass terms" is commonplace when studying Penrose limits of backgrounds with a mass scale [27, 40] but it has appeared before in string theory (see for example [8, 9]).

Here and in section 4 we investigate such questions of perturbative stability. Three kinds of such stability are addressed: classical perturbative stability in sections 3.1 and 3.2, perturbative stability of quantum fields in sections 3.3 and 3.4, and stability of string theory at the level of free strings in section 4. In none of these senses is stability proven, but in each sense it is shown that modes with light-cone tachyons can be stable. That is, we show that the light cone tachyons mentioned above are not, in this case, directly connected to the question of stability. In each case the focus is on homogeneous plane waves, which have a translation symmetry in x^+. Note that in stringy such cases the dilaton will be constant so that the string and Einstein frames differ only by a constant rescaling.

In the current section we investigate plane wave backgrounds in the field theory approximation. To be specific, we assume that the spacetime (1.1) is a classical solution to some gravitating theory and consider small perturbations about this background. For definiteness, we assume that the perturbations can be modeled by massless scalars. Below, we consider the case where the matrix μ_{ij}^2 has at least one negative eigenvalue. The weak energy condition then implies that μ_{ij}^2 also has at least one positive eigenvalue.

3.1 Massless classical field theory on the plane-wave spacetime

Let us take a moment to put our problem into perspective. Our real concern is the stability of the spacetime. This concern is raised by the negative eigenvalues of μ_{ij}^2, which for example appear as negative mass-squared terms in the light-cone string sigma-model. One might therefore think that our spacetime has tachyonic perturbations. But such perturbations can be identified in a classical stability analysis. As a result, one would expect

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this singularity. This effect causes a diverging relative acceleration between two parts of some object. However, the actual distortion of the object (e.g., the change in relative position of these parts) is a second integral of the acceleration and may in some cases remain finite. As a result, one sometimes makes the (diffeomorphism invariant) distinction between 'strong' and 'weak' singularities [37] based on whether an object approaching this singularity is infinitely distorted or is distorted only by a finite amount. Which occurs here depends on whether ξ itself diverges, which in turn is determined by the rate at which μ^2 diverges.

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that the most important features can be seen directly in a classical analysis of linear fields propagating on the plane wave background. One would also expect such perturbations to be qualitatively similar for all massless fields. Indeed, while we will return to quantum effects in section 3.3 and to string effects in section 4, in the end we will see that the most important features are captured by the classical massless scalar field which we investigate below.

Thus we consider a minimally coupled massless scalar field. Such a field satisfies the equation of motion

$$\nabla^2 \phi = 0\,, \tag{3.1}$$

where ∇^2 is the Laplace-Beltrami operator associated with the string metric. Recall that the curvature scalar R and the squared field strength F^2 of any gauge field both vanish in our background and that the dilaton is constant. Thus, dilaton perturbations also satisfy (3.1), as do scalar fields with arbitrary curvature couplings. Furthermore, the equation of motion takes form (3.1) in both the string and Einstein frames. Perturbations of other supergravity fields are discussed in [28], where it is shown that their equations of motion take a similar form in an appropriate gauge.

For the metric (1.1), the wave equation (3.1) is proportional to the condition

$$[-4\partial_- \partial_+ + (x^i \mu_{ij}^2 x^j)\partial_-^2 + 4\delta^{ij}\partial_i\partial_j]\phi = 0\,. \tag{3.2}$$

It is convenient to expand ϕ in a basis of modes. Since μ_{ij}^2 is independent of x^+, we may choose modes $\phi_{p_-,p_+,\beta}$ which satisfy

$$\partial_- \phi_{p_-,p_+,\beta} = ip_- \phi_{p_-,p_+,\beta}\,, \qquad \text{and} \qquad \partial_+ \phi_{p_-,p_+,\beta} = ip_+ \phi_{p_-,p_+,\beta}\,. \tag{3.3}$$

Here, β is any index that labels the remaining degeneracy of the modes. If desired, explicit solutions are readily obtained in terms of Hermite polynomials.

The central question is whether such perturbations are stable. In other words, we must ask whether there can be a complete set of modes $\phi_{p_-,p_+,\beta}$ with real p_+, p_-. The fact that the 'mass term' $(x^i \mu_{ij}^2 x^j)$ in (3.2) can be negative suggests that complex frequencies may be required. Thus, it is worth investigating the matter in some detail.

Unfortunately, the problem is simply not well-defined as currently stated. This is because, as first shown in [41], plane wave spacetimes are not globally hyperbolic. This is the result of an AdS-like property which allows signals to propagate from arbitrary x^i to the transverse origin ($x^i = 0$) in an interval of Δx^+ no larger than π/μ_1, where μ_1^2 is the largest eigenvalue of μ_{ij}^2. As a result, dynamics in plane wave backgrounds becomes well-defined only after a choice of boundary conditions. We will make a particular choice of boundary conditions below and argue that it is a natural one. In principle, other choices of boundary conditions may lead to instabilities not seen in our analysis. However, it will be clear from the nature of our arguments that other simple boundary conditions in fact lead to the same conclusions.

We introduce what we call 'transparent' boundary conditions as follows. First, let us consider an infra-red cut-off theory with a boundary at the surface $x^2 = L^2$. In this cut-off theory we impose the boundary conditions

$$\phi(x^+,x^-,x) = \phi(x^+,x^-,-x)\,, \qquad \partial_r\phi(x^+,x^-,x) = -\partial_r\phi(x^+,x^-,-x) \tag{3.4}$$

at the timelike surface $x^2 = L^2$, where $r^2 = \sum_i (x^i)^2$. We then define the full dynamics by taking the limit $L \to \infty$.

We are led to believe that such boundary conditions are physically reasonable by considering the simple BFHP case $\mu_{ij}^2 = \mu^2\delta_{ij}$ with $\mu^2 > 0$. In particular, it is known [44] that the BFHP spacetime can be conformally embedded in the Einstein static universe ($S^9 \times R$), where it covers all but a single null line which forms the boundary. For conformally coupled scalars, it is clear that the large L limit of (3.4) is equivalent to defining the evolution on the plane wave by conformal mapping initial data to the Einstein static universe, evolving in the ESU without assigning any special status to the null line which forms the conformal boundary of the BFHP spacetime, and then conformally mapping back to the plane wave.[2] Since the Ricci scalar vanishes in the plane wave, we may take massless scalars with any curvature coupling to have the same dynamics.

Now that we have fixed boundary conditions, our problem becomes well defined. However, it is still difficult to explicitly identify a Cauchy surface. We note that the problem simplifies somewhat when the infra-red cut-off described above is imposed. Thus we restrict $x^2 < L^2$. Next, we choose some $\epsilon \ll (L^2|\tilde\mu^2|)^{-1}$, where $\tilde\mu^2$ is the largest negative eigenvalue of μ_{ij}^2. The surface $x^+ = -\epsilon x^-$ is spacelike in the region $x^2 < L^2$ and, given the boundary condition at $x^2 = L^2$, forms a Cauchy surface Σ_ϵ for this region. It is convenient to introduce a coordinate $z = -x^+ + \epsilon x^-$ along the surface and a coordinate $t = x^+ + \epsilon x^-$ transverse to the surface. The associated momenta are $p_z = -\frac{1}{2}p_+ + \frac{1}{2\epsilon}p_-$ and $p_t = \frac{1}{2}p_+ + \frac{1}{2\epsilon}p_-$.

The most transparent analysis would now be to consider modes with arbitrary real p_z and which correspond to a complete set of modes in the x^i directions. One would like to see if such modes lead to real frequencies[3] p_t. However, the form of the equation of motion (3.2) makes this technically difficult.

As a result, we pursue a different strategy here. While we will neither prove or disprove stability, we do show that modes with negative two-dimensional effective squared mass, $H_{p_-} = -(x^i\mu_{ij}^2x^j)p_-^2 + 4\delta^{ij}\partial_i\partial_j$, associated with the $x^\pm$ plane can nevertheless remain stable. These are just the modes which originally motivated our search for instabilities. Note that his set includes modes with long wavelengths in the x^i directions.

3.2 An argument for stability

We begin by restricting to modes with real p_x and p_t, so that p_- is also real. Though we have now assumed stability, the point as outlined above will be to demonstrate that

[2]Note that it is essential that the plane wave map to a set of measure one within the Einstein static universe. Thus, there is no analogue of such transparent boundary conditions for other common conformally flat spacetimes (Minkowski, dS, AdS) which can be embedded in the ESU, as information which flows out through the boundary of, e.g., AdS space will not reenter the spacetime.

[3]In principle, one could also allow exponentially decaying modes. However, the time-reflection symmetry $(x^+,x^-) \to (-x^+,-x^-)$ of our spacetime and boundary conditions guarrantee that exponentially damped modes can arise only if there are counterpart growing modes that would destroy stability.

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the class of stable modes is extremely inclusive. In particular, we will show that the most obvious candidates for instabilities, those having the scalar field analogue of light-cone tachyons, are in fact stable. We continue to consider the cut-off theory described above, in which the boundary conditions 3.4 are imposed at $x^2 = L^2$. Note that such boundary conditions cannot affect local dynamics in small regions of spacetime near the center of the plane wave.

In this setting, the operator $H_{p_-} = -(x^i \mu^2_{ij} x^j) p_-^2 + 4\delta^{ij}\partial_i\partial_j$ is self-adjoint for each real p_- and its eigenstates yield a complete set of modes in the x^i directions. The boundary conditions guarantee that the spectrum of H_{p_-} will be discrete and that it is meaningful to follow the nth eigenstate as a function of p_-. Thus, we may introduce the eigenvalue $E_n(p_-)$ corresponding to this state, and we may use n as the additional label β introduced above.[4]

The label p_+ becomes redundant as a mode $\phi_{p_-,p_+,n}$ satisfies

$$[4p_-p_+ + E_n(p_-)]\phi_{p_-,p_+,n} = 0\,, \tag{3.5}$$

determining p_+ in terms of p_-, n. Note that if E_n did not depend on p_-, this contribution would act like a two-dimensional mass term in (3.5). The dependence on p_- means that we have instead arrived at a non-standard dispersion relation whose analysis will require more care.

Let us therefore estimate $E_n(p_-)$ in various regimes. We will assume that the boundary conditions do not make a large contribution. For small $|p_-|$, the action of H_{p_-} on the nth eigenstate is dominated by the kinetic term, $4\delta^{ij}\partial_i\partial_j$. Given the infrared cutoff L, we must have

$$E_n \sim \frac{\pi^2 n^2}{4L^2} > 0\,, \qquad \text{for } p_-^2|\mu^2|L^4 \ll n^2\,. \tag{3.6}$$

On the other hand, for fixed n, μ^2, L and large p_- the potential term will dominate and we will find

$$E_n \sim -p_-^2|\tilde{\mu}^2|L^2\,, \qquad \text{for } p_-^2|\mu^2|L^4 \gg n^2\,, \tag{3.7}$$

where $\tilde{\mu}^2$ is again the most negative eigenvalue of μ^2. From (3.7) we see that the effective two-dimensional mass is negative for large p_-. Since H_{p_-} depends smoothly on p_-, one expects $E_n(p_-)$ to interpolate smoothly between these two regimes. It will be important in our analysis that E_n is positive for small p_-, as small momentum is the regime where tachyons are most dangerous.

Comparing these results with (3.5), we see that p_+ is linear in p_- for large p_-, but satisfies the dispersion relation for a particle of *positive mass* for small p_-. Graphs of dispersion relations with these properties are sketched in figure 1 below for large and small values of $L^2\tilde{\mu}^2$. The relations shown are plausible ones which match the limiting behaviors described above. They show that these two regimes may be connected in a manner such that the tangent to the dispersion relation is always spacelike, so that the group velocity of waves is less than one.

[4] In non-generic cases a further discrete label will be required to remove a remaining degeneracy.

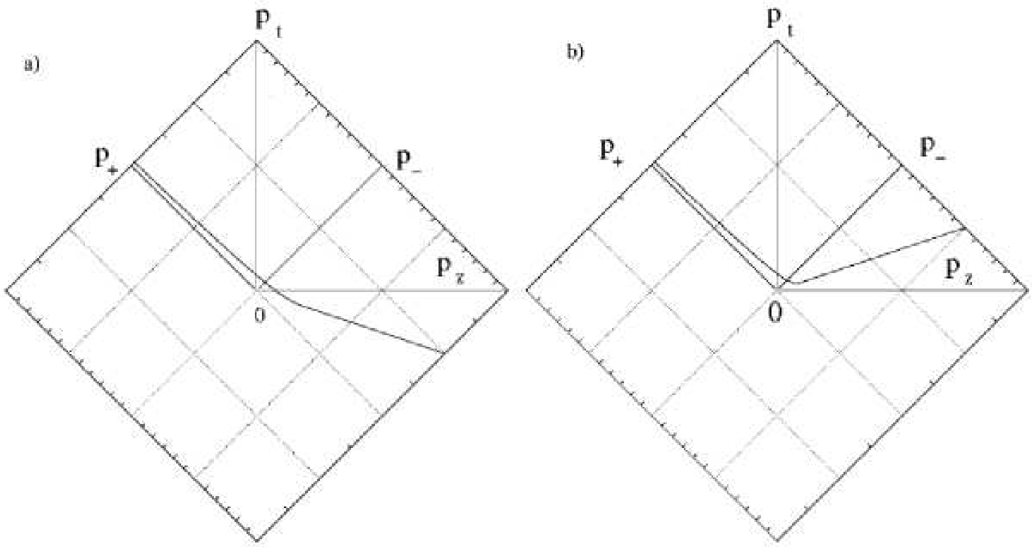

Figure 1: Plots of plausible $p_- > 0$ dispersion relation for a) large $L\tilde{\mu}$ and b) small $L\tilde{\mu}$. Here $\epsilon = 1$. The $p_- < 0$ branches can be obtained by inverting these diagrams through the origin.

Most fundamentally, however, such dispersion relations provide real values of p_t for each p_z. While this is illustrated by the sample dispersion relations shown in the diagrams, it clearly holds independently of the shape of the dispersion relation in the transition region. All that is needed for this conclusion is the asymptotic behavior of the curve and the fact that the eigenvalues $E_n(p_-)$ must be continuous functions of p_-.

Thus, given p_z and n, there are unique associated real values of p_- and p_+ for which the nth eigenstate of H_{p_-} defines a stable mode with frequency p_+. This verifies the claims made at the end of the previous subsection. If the transverse Hilbert space space were finite dimensional, it would constitute a proof of stability. In the infinite dimensional context, however, one must grapple with the technical issue of whether the (p_z, n) modes form a complete set of states on some Cauchy surface.[5] This latter question lies beyond the scope of our current work.

3.3 Quantum field theory

We now turn to a somewhat different notion of stability. It was argued above that the classical field may well be stable. Assuming that this is the case, we now show that no further difficulties arise when the field is quantized. In particular, the vacuum of the free field theory is stable subject to the assumption that there is a complete set of states labeled by real eigenvalues p_-. Much of this material has appeared before in [50, 8, 9], but we repeat it here for clarity and completeness.

[5]Note that this is not equivalent to, and is in fact much more subtle than, proving that these same modes form a complete set of states on the surface $x^+ = constant$; i.e., in $L^2(dx^- d^8x)$. This latter statement follows directly once the states are labeled by the pair (p_-, n), as this problem separates nicely into Fourier analysis in x^- and spectral analysis of H_{p_-}.

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The modes (3.3) are associated with an expansion of the field operator[6]

$$\phi = \int dp_- dp_+ d\beta \, a_{p_-,p_+,\beta} \phi_{p_-,p_+,\beta} \,, \tag{3.8}$$

where p_- ranges over $(-\infty, +\infty)$ and $a_{p_-,p_+,\beta}$ is the requisite operator valued coefficient for the mode $\phi_{p_-,p_+,\beta}$.

Some coefficients $a_{p_-,p_+,\beta}$ will be creation operators and some will be annihilation operators as determined by the commutation relations. As usual, these commutation relations may be computed from the Klein-Gordon inner products of the corresponding mode functions (see, e.g., [51]):

$$[a_{p_-,p_+,\beta}, a_{\tilde{p}_-,\tilde{p}_+,\tilde{\beta}}] = -i(\phi_{p_-,p_+,\beta}, \phi_{\tilde{p}_-,\tilde{p}_+,\tilde{\beta}})_{KG} = -i \int_\Sigma \phi_{p_-,p_+,\beta} \overleftrightarrow{\partial}_n \phi_{\tilde{p}_-,\tilde{p}_+,\tilde{\beta}} \sqrt{g_\Sigma} d\Sigma \,. \tag{3.9}$$

Our convention is that the Klein-Gordon inner product is linear in both entries and that the inner product of real functions is real. In (3.9), n is the unit normal vector to the Cauchy surface Σ and g_Σ is the determinant of the induced metric on Σ. Since the Klein-Gordon product is conserved, (3.9) may be evaluated on any Cauchy surface Σ.

It is well known that the treatment of plane wave backgrounds simplifies in a light cone formalism. This suggests that we take the surface Σ to be of the form $x^+ =$ constant. However, there are several subtleties with this choice. The first is that, as noted above, such surfaces are not Cauchy surfaces. However the only causal curves which fail to cross such a surface are those that asymptotically have constant x^+. Such curves are a set of measure zero. In particular, each null geodesic crosses Σ exactly once unless it lies along an orbit of $\frac{\partial}{\partial x^-}$ (see, e.g., section 2.2).

The other two issues are of a more direct practical sort. Since a surface with $x^+ =$ *constant* is null, the normal cannot be normalized and the induced measure vanishes. However, these problems cancel against each other as $d\Sigma\sqrt{g_\Sigma}\partial_n$ has a well-defined limit when Σ is approximated by a family of spacelike surfaces Σ_λ. We choose Σ_λ to be defined by $x^+ = f_\lambda(x^-, x^i)$ for, e.g.,

$$f_\lambda = \frac{\tanh(x^-/\lambda)}{1 + \lambda^{-2}(x^i \mu_{ij}^2 x^j)^2} \,. \tag{3.10}$$

It is not hard to show that Σ_λ is spacelike for sufficiently large λ, and that the family converges to the surface $x^+ = 0$.

A straightforward calculation shows that in this limit we have $\sqrt{g_\Sigma} \sim \lambda^{-1/2}$, while $\partial_n \sim \lambda^{1/2}\partial_-$ and $d\Sigma\sqrt{g_\Sigma}\partial_n$ asymptotes to a constant times $dx^- d^8x\partial_-$. As a result, the Klein-Gordon inner product on Σ takes the form

$$(\phi_{p_-,p_+,\beta}, \phi_{\tilde{p}_-,\tilde{p}_+,\tilde{\alpha}})_{KG} = A \int_\Sigma \phi_{p_-,p_+,\alpha} \overleftrightarrow{\partial}_- \phi_{\tilde{p}_-,\tilde{p}_+,\tilde{\beta}} dx^- d^8x \,. \tag{3.11}$$

where A is a positive normalization coefficient.

[6]The attentive reader will observe that, despite labeling the modes by p_+, we do not actually make use of the translation symmetry in x^+ anywhere in this section. Thus, the results of this section generalize readily to the case where the wave is not homogeneous. However, this generalization plays no direct role in our discussion of stability.

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Thus, the Klein-Gordon inner product will be proportional to p_-, and p_- will determine the all important sign of the commutation relations. Let us make the following standard choices for the mode functions $\phi_{p_-,p_+,\beta}$. First, we take modes with opposite p_-, p_+ and the same β to be complex conjugates, $\phi_{-p_-,p_+,\beta} = \overline{\phi_{p_-,p_+,\beta}}$. Second, we take modes with distinct β to be orthogonal[7] in $L^2(\mathbb{R}^8, d^8x)$. Finally, taking the normalization of our modes to be $1/\sqrt{A|p_-|}$ in $L^2(\mathbb{R}^9, d^8xdx^-)$, we find that the Klein-Gordon inner product for modes with $p_- > 0$ yields:

$$[a_{p_-,p_+,\beta}, a_{\tilde{p}_-,p_+,\beta}] = \mathrm{sign}(p_-)\delta(p_- + \tilde{p}_-)\delta(p_+ + \tilde{p}_+)\delta(\beta, \tilde{\beta})\,. \tag{3.12}$$

This is the characteristic algebra of creation and annihilation operators. Note that the reality of p_- was critical in writing $p_-/|p_-| = \mathrm{sign}(p_-)$.

From (3.12) we see that $a_{p_-,p_+,\beta}$ is an annihilation operator for $p_- > 0$ and a creation operator for $p_- < 0$. The details of modes with exactly $p_- = 0$ are a mere matter of convention since there are no normalizable $p_- = 0$ modes. Note that when an eigenvalue of μ^2 is negative, the spectrum of $[4\delta^{ij}\partial_i\partial_j - (x^i\mu^2_{ij}x^j)p_-^2]$ covers the entire real line so that negative values of p_+ can arise for any p_- and, in particular, mode functions associated with creation operators can have either sign of p_+. In this sense we will necessarily have negative energy particles. However, we repeat that p_+ always remains real so that the modes of ϕ are stable. The structure of the Fock space is much like that of a massless free field in Minkowski space. In particular, the vacuum is the unique normalizable state with $p_- = 0$. This rules out any pair creation of particles from the vacuum so that the vacuum is stable. Since the theory is non-interacting, each n-particle state is stable as well.

3.4 Interactions in quantum field theory

Suppose, however, that we now turn on some self-interaction for our massive field. The vacuum must remain stable for the reason stated in section 3.3 above: it is the only normalizable state with $p_- = 0$. However, a discussion of the particle states is more interesting.

In Minkowski space, any massless particle remains stable because energy-momentum conservation forbids its decay. But energy-momentum conservation in our background is more subtle. While each surface of constant x^+ is homogeneous, the generators of those translations do not commute with ∂_+. In particular, the momenta $p_i = -i\frac{\partial}{\partial x^i}$ corresponding to simple translations in x^i do not generate symmetries and are not conserved.

This breaking of translation invariance should in general allow any one-particle state to decay. To orient ourselves, we begin by considering those μ^2_{ij} which are positive definite. Here the momenta $p_\pm$ are positive and p_+ has a discrete spectrum. In fact, the mode with lowest p_+ has $p_+ > 0$. As a result, a particle in this lowest mode p_+ is again clearly stable by conservation of p_+. This is also true of strings in the maximally supersymmetric plane wave [25].

In contrast, when μ^2_{ij} has a negative eigenvalue, the spectrum of p_+ is unbounded below and energy-momentum conservation allows any particle to decay. Furthermore, any

[7] Note that orthogonality in $L^2(\mathbb{R}^8, d^8x)$ is preserved under evolution in x^+ since the light cone hamiltonian is self-adjoint in this space for each value of p_-.

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decay products will be dispersed by the repulsive tidal forces removing any hope of stability being restored by interactions among these products. It is clear that an infinite number of particles will be created as $x^+ \to \infty$. As a result, the S-matrix will be non-unitary and the usual scattering theory is inappropriate.[8] However, there remains the possibility of the existence of a self-adjoint hamiltonian which implements unitary time evolution or some other well-defined notion of time evolution over *finite* periods of time. We argue that this is the case in two ways: first by imposing an infra-red cut-off, and second by analogy with a more familiar system with similar properties.

1. One can impose an infra-red cut-off by placing the system in a box. This makes the spectrum of p_+ both bounded below and discrete. As a result, the lowest one-particle states are stable. Since the infra-red regulated theory is sufficient to describe the local dynamics of the full theory within small regions of spacetime, this local dynamics is well-defined.

2. A useful analogy can be constructed by making use of the observation that our Fock space resembles that of a massless field in Minkowski space. In that case, for any interaction, one-particle states are prevented from decay by energy-momentum conservation. But suppose that we break translation invariance by adding a localized potential (a 'scattering center') near the origin. The scattering center can remove excess momentum so that a massless particle is now allowed to decay. Note that massless particles away from the scattering center are stable, so the theory remains unitary for any finite number of scattering centers. However, if we construct an infinite lattice of scattering centers, then the decay process can continue ad infinitum and one expects the theory to become non-unitary. Nonetheless, the theory is locally equivalent to the earlier example with a single scattering center. Thus the model is well-defined as a local quantum field theory.

In fact, non-unitarity due to infra-red divergences is common place even in *free* quantum field theory on non-compact spaces. For example, this feature arises whenever the background is both time dependent and spatially homogeneous with non-compact spatial slices. The time dependence leads to particle creation, which by translation invariance must yield some finite particle number per unit volume. Since space is infinite in volume, the total particle number diverges and the evolution is non-unitary. This is one reason why quantum field theory in general curved backgrounds is often discussed in terms of algebraic local quantum field theory (see, e.g., [52]). Note that this sort of non-unitary has nothing to do with the black hole 'information loss paradox'.

One may ask what are the physical effects of the infinite decay of particles in the plane wave background. For example, does it lead to a large back-reaction on the background spacetime? In the limit of weak coupling, the production rate must be slow. Thus, we

[8] This tendency to decay infinitely was briefly discussed in [27]. It was also noticed that for some string backgrounds the region for which μ_{ij}^2 is negative allows a dual description in which μ_{ij}^2 is positive. One may take this as evidence that the underlying dynamics is well-defined, though typically issues remain associated with objects which are now non-perturbative in the new dual description.

expect that, for weak coupling and if considered for a short time, the particle production instability does not lead to a large back-reaction. It seems clear that, although scattering theory may not be applicable, there is no reason to regard our spacetime as leading to an ill-defined quantum field theory.

4. String theory in the plane-wave background

Since our initial motivation for studying plane waves was their importance as exactly solvable string backgrounds, it is important to ask how stringy effects modify the quantum field theoretic picture described above. Of particular concern is the fact that the eigenvalues of μ_{ij}^2 are often described as leading to mass terms in the string sigma model, so that one might expect negative eigenvalues to lead to tachyons and instabilities. We shall argue this is not the case, though some interesting behavior does result.

We begin by briefly reviewing the quantization of the string in an arbitrary time-independent plane wave background. We use the Brinkmann coordinates x^μ (so that the spacetime metric again takes the form 1.1) and choose worldsheet coordinates (τ, σ) with τ an (as yet unknown) function of x^+, and σ an orthogonal spacelike coordinate normalized to have period 2π. The remaining gauge freedom is fixed by requiring the worldsheet metric to be conformal to the Minkowski metric. The function $\tau(x^+)$ is then determined by the equations of motion.

Varying the sigma-model action with respect to x^- leads to the equation

$$(-\partial_\tau^2 + \partial_\sigma^2)x^+ = 0\,. \tag{4.1}$$

Note that the only solutions consistent with the gauge fixing specified above are $x^+(\tau) = -\frac{1}{2}p_-\tau + x^+(0)$ for some constants p_- and $x^+(0)$.

For simplicity, let us assume that the matrix μ_{ij}^2 is diagonal, with eigenvalues μ_i^2. The equation of motion for a transverse coordinate x^i is then

$$(-\partial_\tau^2 + \partial_{\sigma-}^2)x^i - \frac{1}{4}p_-^2\mu_i^2 x^i = 0\,, \tag{4.2}$$

so that the eigenvalue μ_i^2 contributes a squared-mass to the field x^i. While μ_i^2 itself can be rescaled by rescaling x^+ and x^-, the factor of p_-^2 scales in the opposite way so that $p_-^2\mu_i^2$ is invariant under this transformation.

As usual, it is beneficial to decompose the coordinates x^i into modes. We refer to the mode of x^i with σ-dependence $e^{im\sigma}$ as x_m^i. Taking the Fourier transform of equation (4.2) yields

$$\partial_\tau^2 x_m^i = \left(m^2 + \frac{1}{4}p_-^2\mu_i^2\right)x_m^i\,, \tag{4.3}$$

so that the mode is stable for $m^2 + \frac{1}{4}p_-^2\mu_i^2 > 0$ and unstable for $m^2 + \frac{1}{4}p_-^2\mu_i^2 < 0$. In the latter case, gravitational tidal forces cause the string to stretch indefinitely in the x^i direction.

Due to the compactness of the string, it is clear that only a finite number of modes will be unstable for any given values of p_-, $\mu_i^2 < 0$. Suppose first that $0 > \frac{1}{4}p_-^2\mu_i^2 > -1$,

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which, roughly speaking, is the case where the spacetime curvature is much less than string scale in the reference frame of the string.[9] Since only the zero mode is unstable, the string has no problem retaining its integrity. The only instability is that the center of mass of the string is rapidly pushed away from $x^i = 0$. One expects that the string spectrum will be discrete, and that different internal excitations can be interpreted as particles corresponding to different quantum fields. The stability of such fields has already been discussed in section 3.

More generally, we may work out the canonical momenta to find that the squared masses of string states are given by

$$M^2 = p_\alpha p_\beta g^{\alpha\beta} = \sum_{m\geq 0} \frac{1}{4} p_-^2 [\overline{x}^i_m \mu^2_{ij} x^j_m] + \sum_{m>0} (\overline{p}^i_m \delta_{ij} p^j_m + m^2 \overline{x}^i_m \delta_{ij} x^j_m), \tag{4.4}$$

where $\overline{x}^i_m$ represents the complex conjugate of x^i_m. Similarly, the dispersion relation takes the form

$$p_+ p_- = \sum_{m\geq 0} \left(\frac{1}{4} p_-^2 [\overline{x}^i_m \mu^2_{ij} x^j_m] + \overline{p}^i_m \delta_{ij} p^j_m + m^2 \overline{x}^i_m \delta_{ij} x^j_m \right), \tag{4.5}$$

One sees that the right hand side of either (4.4) or (4.5) can easily be negative. However, this occurs only at large p_-. The dispersion relation is qualitatively of the same form as that in section 3, where we saw that it did not in fact lead to instabilities. Note that while there is now a new conceptual point associated with the fact that the locally measured M^2 is a function of momentum (whereas in section 3 the combination $g^{\alpha\beta} p_\alpha p_\beta$ was constant), this does not in any way effect the analysis of the dispersion relation (4.5).

As a result, the arguments given in section 3 apply here as well. Said differently, while there are clearly *particle* instabilities in which the strings are infinitely stretched or pushed off to infinity, there appear to be no *field theoretic* instabilities in which small field perturbations grow exponentially in x^+.

However, an interesting phenomenon arises when one investigates the low energy effective description of strings in this background.[10] One might expect that a low energy limit could be obtained by restricting the light cone energy (p_+) to be small. However, since negative p_+ states exist, interactions will generate particles with arbitrarily large p_+ even when the total value of p_+ is small. The same is true if we attempt to restrict to small p_t for for any t associated with one of the cut-off 'Cauchy surfaces' Σ_ϵ of section 3.1. In contrast, we have seen that p_- is positive for all particles, so that it makes sense to restrict to small p_-. While only a finite number of fields can then reach negative p_+, the associated low energy theory nevertheless contains an infinite number of fields.

For completeness, let us briefly discuss the fermionic sector. Following [20, 21, 54] the fermionic part of the string action can be written in terms of the pull back to the world

[9] This interpretation follows by using (2.10) and the appropriate Killing field ξ to write $p_-^2 \mu_i^2 = p_-^2 R^i_{+i+}$ and recalling the relation $p_- = (\frac{\partial x^+}{\partial \tau})$.

[10] In principle, one might wonder if the analysis of section 3.3 applies to effective fields with the non-standard kinetic terms associated with the non-standard dispersion relation (4.4). That no problem arises is suggested by the similarity of (4.5) and (3.5), or may be verified directly in light-cone gauge. This is particularly straightforward if one uses a covariant quantization method such as that of Peierls [53].

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sheet of the covariant derivative entering in the variation of the gravitino:

$$D_a = \partial_a + \frac{1}{4}\partial_a x^m \left[\left(\omega_{\mu\nu m} - \frac{1}{2}H_{\mu\nu m}\rho_3\right)\Gamma^{\mu\nu} + \left(\frac{1}{3!}F_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\rho_1 + \frac{1}{2\cdot 5!}F_{\mu\nu\lambda\rho\kappa}\Gamma^{\mu\nu\lambda\rho\kappa}\rho_0\right)\Gamma_m\right] \tag{4.6}$$

where the ρ_s-matrices in the I, J space are the Pauli matrices $\rho_1 = \sigma_1$, $\rho_0 = i\sigma_2$. The action becomes

$$\mathcal{L} = i(\eta^{ab}\delta_{IJ} - \epsilon^{ab}(\rho_3)_{IJ})\partial_a x^m \bar{\theta}^I \Gamma_m D_b \theta^J \tag{4.7}$$

Here θ^I $(I = 1, 2)$ are the two real positive chirality 10-d MW spinors and $\rho_3 = \mathrm{diag}(1, -1)$. Thus the masses of the fermions are completely determined by the value of the RR form fields. In the simplest case of a background with only the 5-form field strength non-vanishing $F_5 = f(x^+)dx^+ \wedge (\omega_4 + *\omega_4)$, the only nontrivial Einstein equation of motion is $\mathrm{Tr}\,\mu_{ij}^2 = [f(x^+)]^2$ which does not directly feel the effect of one negative eigenvalue in μ_{ij}^2. Thus, one may easily deform, for example, the BFHP plane wave to one with a negative eigenvalue without changing the behavior of worldsheet Fermions. As a result, at least for appropriate choices of Ramond-Ramond fields, consideration of Fermions adds nothing new to the picture discussed above.

5. Discussion

In section 2 we have analyzed the singularity structure of plane wave metrics and provided a simple criterion to determine the presence of singularities. In appendix A we have also presented careful analysis of the Penrose limit of an AdS background dual to a gauge theory having a mass scale. Our analysis provides a strictly gravitational approach to this problem.

We have also begun to explore the stability of classical and quantum fields in backgrounds having eigenvalues of μ_{ij}^2 of mixed signs.[11] We have reduced the problem to proving completeness of a natural set of modes. An important remark is that this set of stable modes *does* include all obvious candidate modes for instabilities. In addition, our results would imply stability if the transverse Hilbert space were of finite dimension. Thus, we have reduced the problem to a rather technical question of operator spectra. While we have given only the most cursory treatment of boundary conditions, we provided some justification that this is sufficient based on the analyses of plane wave asymptotic structures presented in [44, 43].

For completeness, some comments are required on the relation of our results to earlier work concerning 'sandwich waves,' in which μ_{ij}^2 vanishes before some initial x_i^+ and after some final x_f^+. First, reference [35] claims that in field theory one finds a large back-reaction for any excited state due to a focusing of the field associated with the focussing of geodesics described in section 2. However, this apparent problem is readily seen to be an artifact of the construction used in [35] and to disappear when appropriate wave packet states are used. For example, in the homogeneous plane wave it is natural to choose states adapted

[11] While we have not addressed the negative definite case directly, homogeneous such plane waves violate all positive energy conditions and are unphysical.

to the x^+ translation symmetry which have purely exponential dependence on x^+. Such states clearly do not focus to infinite strength so that back-reaction can be small.

Finally, we have discussed certain features of string theory in such plane wave backgrounds. Again, the theory appears to be stable, though any low energy description necessarily involves an infinite number of fields.

We have not yet considered the implications of string interactions. Let us take a moment to do so now. Such interactions trigger an instability of one-particle states in parallel with that discussed in section 3.4 and the resulting S-matrix will be non-unitary. As in the field theoretic case, the issue might be resolved through a more local treatment. However, it is an open question to what extent string theory may be formulated locally and the difficulties of, for example, using a string field approach to closed strings are well known.

One difference between the string and field theoretic cases is the large number of stringy negative energy states that exist at large p_-. As we have seen, imposing a strict infra-red cutoff on x^i and an ultra-violet cutoff on p_- discards most of these states and truncates the system to having only a finite number of negative energy fields. However, as this cut-off is taken to infinity the number of negative energy fields grows exponentially. One expects this to have important effects, at the very least on the thermodynamics of the system. More fundamentally, it raises the possibility that non-locality at the string scale could interact with these negative energy states in a disastrous fashion.

On the other hand, it is also possible that string interactions act to limit the instability discussed above. Most of the negative energy modes require large p_- but, since p_- is positive and conserved, finite coupling will induce strings with large p_- to decay into states with small p_-; i.e., into strings which are more stable. It would therefore be interesting to understand a strong coupling description of this effect.

There are thus many interesting and fundamental issues raised by string interactions in negative eigenvalue plane wave backgrounds. A proper analysis would seem to require several new techniques, possibly in a string field theoretic setting. We hope that the questions raised by these solvable backgrounds will spur the development of such techniques.

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A. Penrose limit of Schwarzschild black hole in AdS

Taking the Penrose limit of supergravity backgrounds dual to gauge theories with a mass scale has been one of the most natural generalizations of the BMN construction. In most cases intuition from the field theory side has helped in clarifying the effect of the Penrose limit on the original background. However, a universal understanding of the Penrose limit in intrinsically gravitational terms is lacking in the recent literature. Here we elaborate on a concrete example[12] — the Schwarzschild black hole in AdS. Our intention is to provide a careful interpretation of the effect of the limit on the original background along the lines of the work by Geroch [39]. This approach was implicitly assumed also in [19].

The metric we consider is

$$\begin{aligned} ds^2 = &-\left(\frac{r^2}{R^2}+1-\frac{M}{r^2}\right)dt^2+\left(\frac{r^2}{R^2}+1-\frac{M}{r^2}\right)^{-1}dr^2+r^2d\Omega_3^2+ \\ &+R^2\left[\cos^2\theta d\psi^2+d\theta^2+\sin^2\theta d\tilde{\Omega}_3^2\right]. \end{aligned} \tag{A.1}$$

The above metric describes the Schwarzschild black hole in AdS in global coordinates. To perform the Penrose limit we consider a null geodesic in the equitorial 3-plane (t, r, ψ) and located at $\theta = 0$. Here, ψ is the angle around the great circle of S^5. As a result, ∂_ψ is a Killing field.

The effective lagrangian that is equivalent to solving the geodesic equations is:

$$\mathcal{L} = -\left(\frac{r^2}{R^2}+1-\frac{M}{r^2}\right)\dot{t}^2+\left(\frac{r^2}{R^2}+1-\frac{M}{r^2}\right)^{-1}\dot{r}^2+R^2\dot{\psi}^2\,, \tag{A.2}$$

where dot represents derivative with respect to the affine parameter U. The aim is, following Penrose's prescription [22](see also [19]), to find new coordinates (U, V, Y^i) where the metric takes the form

$$ds^2 = dV(2dU+\alpha dV+\beta_i dY^i)+C_{ij}dY^idY^j\,, \tag{A.3}$$

where α, β_i and C_{ij} depend on all the the coordinates. Once the metric is in this form the Penrose limit is given by the scalings

$$ds^2 = \Omega^{-2}ds^2(\Omega)\,,\qquad U = u\,,\qquad V = \Omega^2 v\,,\qquad Y^i = \Omega y^i\,,\qquad \Omega\to 0\,. \tag{A.4}$$

Note that (at $\theta = 0$) ∂_U is a null vector since

$$g_{UU} = G_{tt}\dot{t}^2+G_{rr}\dot{r}^2+G_{\psi\psi}\dot{\psi}^2 = 0 \tag{A.5}$$

by definition of a null geodesic. The equations of motion following from (A.2) are particu-

[12]The Penrose limit of this background was formally considered in [38].

larly simple since the effective lagrangian does not depend explicitly on t or ψ:

$$\dot{\psi} = \frac{\mu}{R^2}\,, \qquad \dot{t} = \frac{E}{\frac{r^2}{R^2} + 1 - \frac{M}{r^2}}\,, \qquad \dot{r}^2 + \frac{\mu^2}{R^4}r^2 - \frac{\mu^2 M}{R^2 r^2} = E^2 - \frac{\mu^2}{R^2}\,. \tag{A.6}$$

These three equations completely determine the dependence of the coordinates (r, t, ψ) on the affine parameter U. To complete the coordinate transformation we use the conditions that $G_{UV} = 1$ and $G_{U\Phi} = 0$. The final coordinate change is the of the form

$$r = r_U\,, \qquad t = t_U - \frac{V}{E} + \mu\Phi\,, \qquad \psi = \frac{\mu}{R^2}U + E\Phi\,, \tag{A.7}$$

where r_U and t_U are functions only of U determined by (A.6). Taking the Penrose limit following (A.4) we find the following metric in Rosen coordinates

$$ds^2 = 2dudv + (E^2R^2 - \mu^2 f(r))d\phi^2 + r^2 ds^2(\mathbb{R}^3) + R^2 ds^2(\mathbb{R}^4) - \frac{\mu^2}{R^2}\vec{z}^2\, du^2\,, \tag{A.8}$$

where $\vec{z}$ parametrizes the $\mathbb{R}^4$ originating from θ and $\tilde{\Omega}_3$, $f(r) = \frac{r^2}{R^2} + 1 - \frac{M}{r^2}$ and r is related to u as

$$2\frac{\mu}{R^2}u = \arctan \frac{\mu r^2/R^2 - E^2R^2/2\mu}{\sqrt{(E^2 - \frac{\mu^2}{R^2})r^2 - \frac{\mu^2}{R^4}r^4 + \frac{\mu^2 M}{R^2}}}\,. \tag{A.9}$$

An arbitrary metric in Rosen coordinates

$$ds^2 = 2dudv + C_{ij}(u)dy^i dy^j\,, \tag{A.10}$$

can be presented in Brinkmann form

$$ds^2 = -4dx^+dx^- + \left[\sum_{i,j} A_{ij}(x^+)x^i x^j\right](dx^+)^2 + \sum_i dx^i dx^i\,, \tag{A.11}$$

by means of the following coordinate transformation:

$$u = 2x^+\,, \qquad v = x^- - \frac{1}{2}\sum_{i,j} M_{ij}(x^+)x^i x^j\,, \qquad y^i = \sum_i Q^i_j(x^+)x^j \tag{A.12}$$

where

$$C_{ij}Q^i_k Q^j_l = \delta_{kl}\,, \qquad C_{ij}(Q'^i_j Q^j_l - Q'^i_k Q^j_l) = 0\,, \qquad M_{ij} = C_{kl}Q'^k_l Q^l_j\,, \tag{A.13}$$

where prime indicates derivative with respect to x^+. In the particular case when $C_{ij} = a_i^2(x^+)\delta_{ij}$ is diagonal we have that

$$A_{ij} = \frac{(a_i(x^+))''}{a_i(x^+)}\delta_{ij} \tag{A.14}$$

For the Schwarzschild black hole we obtain a metric that can be written in Brinkmann coordinates as:

$$
\begin{aligned}
ds^2 = & -4dx^+dx^- - \frac{\mu^2}{R^4}\left[\sum_{a=1}^{4} x_a^2 + \left(1+\frac{MR^2}{[r(x^+)]^4}\right)\sum_{b=5}^{7} x_b^2 + \left(1-3\frac{MR^2}{[r(x^+)]^4}\right)x_8^2\right](dx^+)^2 + \\
& + \sum_{i=1}^{8} dx_i^2\,,
\end{aligned}
\tag{A.15}
$$

where $r(x^+)$ is given implicitly by (A.9) with $u = 2x^+$.

One question that naturally arises when considering Penrose limits of metrics containing dimensionful parameters is what happens to these parameters under the rescaling. In what follows we address this question in the specific case of AdS-Schwarzschild but our approach is universal. First, note that the Penrose limit can be thought of in two parts (performed simultaneously). One is an overall scaling of the metric by Ω^{-2}. In string theoretic applications this is typically interpreted as a change in the fundamental length scale, in which units all quantities are taken to be measured. As a result, dimensionful parameters (R_Ω, M_Ω) computed from the metric at finite Ω are taken to represent the ratio of the true physical parameters to the fundamental scale and so depend on Ω. Dimensional analysis then tells us that we have

$$
R_\Omega = \frac{R}{\Omega} \tag{A.16}
$$

and

$$
M_\Omega = \frac{M}{\Omega^2}\,. \tag{A.17}
$$

Translating this result to the dual field theory, one takes a limit in which $g_{YM}N$ becomes infinitely large in a way that matches R_Ω. This is the content of the statement "$R = \Omega^{-1}$" commonly associated with this limit [25]. In particular, one does not rescale R in addition to taking the Penrose limit; rather, the rescaling of R is a consequence of interpreting the strict Penrose limit in the dual field theory. Tracing through the duality map also tells us on the field theory side how the temperature of the thermal bath corresponding to the black hole varies with Ω. Not surprisingly, one finds that the temperature of the black hole is $T_\Omega = \Omega T$ and therefore goes to zero in the limit as is compatible with our knowledge that the "black hole" disappears (see [42] for a discussion of the possibility of black holes in plane waves).

In much the same way that we defined M_Ω and R_Ω above, one may define E_Ω and μ_Ω by calculating them from (A.6) applied to the favored null geodesic in the spacetime ds^2_Ω. The key point is that the geodesic remains the same in terms of the coordinates $\tilde{t} = t/R, \tilde{r} = r/R$, and ψ and that the parametrization remains unchanged (we always use affine parameter U). (The geodesic also remains unchanged in terms of t, r, ψ, but $\tilde{t}, \tilde{r}$, and ψ are nicer as the rescaled metric maintains the same form except for the replacement of R with R_Ω.) Thus, $\dot{\psi} = \mu/R^2$ is independent of Ω and $\dot{\tilde{t}} = E/f(r)R$ is independent of Ω. In other words, we may identify

$$
E_\Omega = \frac{E}{\Omega} \tag{A.18}
$$

and

$$\mu_\Omega = \Omega^{-2}\mu\,. \tag{A.19}$$

It is natural to assert that such scalings are the supergravity dual of the statement on the field theory side that we are focusing on states with large energy and large R-charge.

To further simplify the above metric we introduce $q = ER/\mu$ and $\tilde{r} = r/(qR)$ and $\tilde{m} = M/(qR^2)$. Then we have

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \left[\sum_{a=1}^{4} x_a^2 + \left(1+\frac{\tilde{m}}{[\tilde{r}(x^+)]^4}\right)\sum_{b=5}^{7} x_b^2 + \left(1-3\frac{\tilde{m}}{[\tilde{r}(x^+)]^4}\right)x_8^2\right]\left(d\frac{\mu x^+}{R^2}\right)^2 + \\ &\quad + \sum_{i=1}^{8} dx_i^2\,. \end{aligned} \tag{A.20}$$

Furthermore, note that the equation (A.9) relating u and r can be written:

$$\frac{\mu x^+}{R^2} = \arctan\frac{\tilde{r}^2 - 1/2}{\tilde{r}^2 - \tilde{r}^4 + \tilde{m}}\,, \tag{A.21}$$

so that the entire final metric is specified by the parameter $\tilde{m}$. It is worth noting that the same background is obtained as the limit of non-extremal D3-branes in Poincare coordinates. Now we have an answer to the question of what happened to the $r = 0$ singularity of the original black hole. We see that $r = 0$ remains a curvature singularity in the Penrose limit.

References

[1] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112].

[2] D.I. Olive, E. Rabinovici and A. Schwimmer, *A class of string backgrounds as a semiclassical limit of WZW models*, *Phys. Lett.* **B 321** (1994) 361 [hep-th/9311081].

[3] K. Sfetsos, *Gauging a nonsemisimple WZW model*, *Phys. Lett.* **B 324** (1994) 335 [hep-th/9311010].

[4] E. Kiritsis and C. Kounnas, *String propagation in gravitational wave backgrounds*, *Phys. Lett.* **B 320** (1994) 264 [hep-th/9310202], addendum *ibid.* **B 325** (1994) 536.

[5] E. Kiritsis, C. Kounnas and D. Lüst, *Superstring gravitational wave backgrounds with space-time supersymmetry*, *Phys. Lett.* **B 331** (1994) 321 [hep-th/9404114].

[6] R. Güven, *Plane waves in effective field theories of superstrings*, *Phys. Lett.* **B 191** (1987) 275.

[7] D. Amati and C. Klimčík, *Nonperturbative computation of the Weyl anomaly for a class of nontrivial backgrounds*, *Phys. Lett.* **B 219** (1989) 443.

[8] G.T. Horowitz and A.R. Steif, *Space-time singularities in string theory*, *Phys. Rev. Lett.* **64** (1990) 260.

[9] G.T. Horowitz and A.R. Steif, *Strings in strong gravitational fields*, *Phys. Rev.* **D 42** (1990) 1950.

JHEP01(2003)076

[10] G.T. Horowitz, *Singularities in string theory*, in *Proceedings of strings '90*, College Station, Texas, March 1990, World Scientific, Singapore 1991.

[11] R.E. Rudd, *Compactification propagation*, *Nucl. Phys.* **B 352** (1991) 489.

[12] H.J. de Vega and N. Sanchez, *Strings falling into space-time singularities*, *Phys. Rev.* **D 45** (1992) 2783.

[13] H.J. de Vega, M. Ramon Medrano and N. Sanchez, *Classical and quantum strings near space-time singularities: gravitational plane waves with arbitrary polarization*, *Class. and Quant. Grav.* **10** (1993) 2007.

[14] A.A. Tseytlin, *String vacuum backgrounds with covariantly constant null Killing vector and 2-d quantum gravity*, *Nucl. Phys.* **B 390** (1993) 153 [hep-th/9209023].

[15] G.T. Horowitz and A.A. Tseytlin, *On exact solutions and singularities in string theory*, *Phys. Rev.* **D 50** (1994) 5204 [hep-th/9406067].

[16] A.A. Tseytlin, *Exact solutions of closed string theory*, *Class. and Quant. Grav.* **12** (1995) 2365 [hep-th/9505052].

[17] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, *J. High Energy Phys.* **08** (2001) 036 [hep-th/0105308].

[18] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[19] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].

[20] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[21] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109];
J.G. Russo and A.A. Tseytlin, *On solvable models of type-IIB superstring in NS-NS and RR plane wave backgrounds*, *J. High Energy Phys.* **04** (2002) 021 [hep-th/0202179].

[22] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity*, Reidel, Dordrecht, 1976;
Techniques of differential topology in relativity, SIAM, 1972.

[23] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[24] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[25] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[26] M.W. Brinkmann, *On Riemann spaces conformal to euclidean spaces*, *Proc. Natl. Acad. Sci. U.S.* **9** (1923) 1, see section 21.5.

[27] E.G. Gimon, L.A. Pando Zayas and J. Sonnenschein, *Penrose limits and RG flows*, hep-th/0206033.

[28] D. Brecher, J.P. Gregory and P.M. Saffin, *String theory and the classical stability of plane waves*, hep-th/0210308.

[29] H. Bondi, F.A.E. Pirani and I. Robinson, *Gravitational waves in general relativity, III. Exact plane waves*, *Proc. Roy. Soc. (London)* **A251** (1959) 519.

[30] J. Hély, *Compt. Rend.* **249** (1959) 1867.

[31] A. Peres, *PP-waves. The original article: some gravitational waves*, *Phys. Rev. Lett.* **3** (1959) 571 [hep-th/0205040].

[32] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in *Gravitation: an introduction to current research*, L. Witten ed., Wiley, New York 1962.

[33] D. Kramer, H. Stephani, E. Herlt, M. MacCallum and E. Schmutzer, *Exact solutions of Einstein's field equations* Cambridge University Press, Cambridge 1979.

[34] C. Misner, K. Thorne and J. Wheeler, *Gravitation*, W.H. Freeman and Co., New York 1971.

[35] J. Garriga and E. Verdaguer, *Scattering of quantum particles by gravitational plane waves*, *Phys. Rev.* **D 43** (1991) 391.

[36] K. Skenderis and M. Taylor, *Branes in AdS and PP-wave spacetimes*, *J. High Energy Phys.* **06** (2002) 025 [hep-th/0204054].

[37] F.J. Tipler, *Singularities in conformally flat spacetimes*, *Phys. Lett.* **A 64** (1977) 8.

[38] L.A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *J. High Energy Phys.* **05** (2002) 010 [hep-th/0202186].

[39] R. Geroch, *Limits of spacetimes*, *Commun. Math. Phys.* **13** (1969) 180.

[40] S. Bhattacharya and S. Roy, *Penrose limit and NCYM theories in diverse dimensions*, hep-th/0209054.

[41] R. Penrose, *A remarkable property of plane waves in general relativity*, *Rev. Mod. Phys.* **37** (1965) 215.

[42] V.E. Hubeny and M. Rangamani, *No horizons in PP-waves*, *J. High Energy Phys.* **11** (2002) 021 [hep-th/0210234].

[43] D. Marolf and S.F. Ross, *Plane waves: to infinity and beyond!*, *Class. and Quant. Grav.* **19** (2002) 6289 [hep-th/0208197].

[44] D. Berenstein and H. Nastase, *On lightcone string field theory from super Yang-Mills and holography*, hep-th/0205048.

[45] R. Geroch, E.H. Kronheimer and R. Penrose, *Ideal points in space-time*, *Proc. Roy. Soc. Lond.* **A 327** (1972) 545.

[46] R. Budic and R.K. Sachs, *Causal boundaries for general relativistic space-times*, *J. Math. Phys.* **15** (1974) 1302.

[47] L.B. Szabados, *Causal boundary for strongly causal space-time*, *Class. and Quant. Grav.* **5** (1988) 121.

[48] *Causal boundary for strongly causal space-time II*, *Class. and Quant. Grav.* **6** (1989) 77.

[49] D. Marolf and S. Ross, in preparation.

[50] G.W. Gibbons, *Quantized fields propagating in plane wave space-times*, *Commun. Math. Phys.* **45** (1975) 191.

JHEP01(2003)076

[51] B. DeWitt, *The spacetime approach to quantum field theory*, in *Relativity, groups, and topology II. Les Houches 1983*, B. DeWitt and R. Stora, ed., part 2, p. 381, North-Holland, New York, 1984.

[52] R. Haag, *Local quantum physics*, Springer, Berlin 1992.

[53] R.E. Peierls, *The commutation laws of relativistic field theory*, *Proc. Roy. Soc. (London)* **214** (1952) 143.

[54] M. Cvetič, H. Lu, C.N. Pope and K.S. Stelle, *Linearly-realised worldsheet supersymmetry in PP-wave background*, hep-th/0209193.